B.Sc Part II (Hons)

Quotient Ring (or factor Ring):

Definition:— Let R be a ring and N be an ideal in R. Then the set R/N whose elements are the set of elements of the form $a+N$ where a is a fixed element in R.

Theorem (A):— Suppose R is a ring, S an ideal of R. Let f be a mapping from R to R/N defined by $f(a) = S+a$ for all $a \in R$.

Then f is a homomorphism of R onto R/S.

OR.

The quotient ring R/S of a ring R with respect to an ideal S is a homomorphic image of R.

Proof:— Let a, b be two elements of R.

Then we have,

$$f(a+b) = S+(a+b) = (S+a)+(S+b)$$
$$= f(a)+f(b)$$

And
$$f(ab) = S+(ab) = (S+a)(S+b)$$
$$= f(a)+f(b)$$

Hence $f: R \to R/S$ is a homomorphism of R onto R/S.

Thus every Quotient ring of a ring is a homomorphic image of the ring.

Theorem (B):— Fundamental Theorem on homomorphism of rings.

Every homomorphic image of a Ring R is isomorphic to some residue class ring (Quotient ring) there of.

Proof:— Let f be a homomorphism mapping of the Ring $(R, +, \cdot)$ onto the ring $(R', \oplus, \odot)$.

Let S be the ideal of R. Then we require to prove that the quotient ring R/S is isomorphic of R'.

Let R' be the homomorphic image of a ring R and f be the corresponding homomorphism. Then f is a homomorphism of R onto R'.

Let S be the kernel of the homomorphism. Then S is an ideal of R. Therefore R/S is a quotient ring relative to S.

We shall prove that $R/S \cong R'$

Let $a \in R$, then $f(a) \in R'$ and $S+a \in R/S$

Consider the mapping $\phi : R/S \to R'$ such that

$$\phi(S+a) = f(a) \text{ for all } a \in R.$$

First we shall show that the mapping ϕ is well-defined i.e. if $a, b \in R$ and $S+a = S+b$, then $\phi(S+a) = \phi(S+b)$.

We have, $S+a = S+b \Rightarrow a-b \in S$.

$\Rightarrow f(a-b) = 0'$ [since S is the kernel of homomorphism and $0'$ is the zero element of R']

$\Rightarrow f[a+(-b)] = 0' \Rightarrow f(a)+f(-b) = 0'$

$\Rightarrow f(a)-f(b) = 0'$

$\Rightarrow f(a) - f(b) \Rightarrow \phi(S+a) = \phi(S+b)$ $\therefore \phi$ is well defined.

ϕ is one-one: We have $\phi(S+a) = \phi(S+b) \Rightarrow f(a)$

$\Rightarrow f(a) - f(b) = 0' \Rightarrow f(a) + f(-b) = 0'$

$\Rightarrow f(a-b) = 0' \Rightarrow a-b \in S$ $\because S$ is kernel of f.

$\Rightarrow S+a = S+b$ $\therefore \phi$ is one-one.

ϕ is onto: — Let y be any element of R'. Then $y = f(a)$ for some $a \in R$, because f is onto R'.

Now $y = f(a) = \phi(S+a)$, $S+a \in R/S$

Therefore ϕ is onto R'. Finally we have

$\phi[(S+a)+(S+b)] = \phi[S+(a+b)] = f(a+b) = f(a)+f(b) = \phi(S+a) + \phi(S+b)$

Also $\phi[(S+a)(S+b)] = \phi(S+ab) = f(ab) = f(a)f(b)$
$= \phi(S+a)\,\phi(S+b)$

$\therefore \phi$ is an isomorphism of R/S onto R'.

Hence $R/S \cong R'$.

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